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INVOLUTIONS ON THE HILBERT SPHERES AND RELATED PROPERTIES IN (I-D) SPACES

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Involutions on the Hilbert spheres and related properties in (I-D) spaces

Raymond Y.T. Wong *)

1.

The purpose of this paper is to study the category whose objects are pairs (X,α), (Y,β), ... where X, Y, ... are spaces equipped with involutions (= period 2 homeomorphisms) α, β, ... and whose morphisms are maps $m: (X,\alpha) \to (Y,\beta)$ of pairs such that m is a map of X into Y which commutes with α, β:

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ \downarrow \alpha & & \downarrow \beta \\ X & \xrightarrow{m} & Y \end{array} .$$

We can speak of m as an imbedding, homeomorphism (isomorphism), etc. If the objects are C^∞-smooth manifolds equipped with C^∞-smooth involutions, then the morphisms will be C^∞-diffeomorphisms.

Our principal results deal with objects which are infinite dimensional Hilbert spheres (spaces) carrying fixed point free involutions and with morphisms which are imbeddings or homeomorphisms. For an account of objects (finite dimensional spaces, maps) see Conner and Floyd [6].

In section 2 we state our main results. Proofs of these will be given in section 7. Section 3 gives some preliminary lemmas which will be needed in sections 5, 6. Related properties in infinite dimensional spaces and their proofs are treated in section 4. In section 5 and 6 we develope key lemmas which will lead to a proof of our main theorems stated. Some open questions are raised in the final section.

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2. Principal results

Theorem 1. *Every fixed point free involution on the Hilbert sphere is conjugate to the antipodal map.*

In this paper by a Hilbert space H we mean an *infinite dimensional Hilbert space* $l_2(N)$ of all square summable real sequences indexed by an infinite abstract set $I(N)$ of cardinality N. We let l_2 denote the separable Hilbert space $l_2(N_0)$. The Hilbert sphere S is defined to be the unit sphere of H. The involution on S which takes each x to -x is called *antipodal*. Let X, Y be homeomorphic ($\cong$) spaces. Denote by G(X,Y) the space of homeomorphisms of X onto Y with the compact-open topology. Write G(X,Y) as G(X) when X = Y. An $f_1 \in G(X)$ is *conjugate* to an $f_2 \in G(Y)$ if there is a $g \in G(X,Y)$ such that $f_2 = gf_1g^{-1}$. It is clear that conjugate is an equivalence relation.

With exactly the same argument we also prove

Theorem 2. *Every fixed point free* C^∞*-smooth involution on the separable Hilbert sphere is smoothly conjugate to the antipodal map.*

Here smoothly conjugate means that the connecting function is also C^∞-smooth. For any Hilbert space $H = l_2(N)$, let

$$H_f = \{x(\cdot) \in l_2(N): \ x(i) = 0 \text{ for almost all } i \in I(N)\}$$

and

$$S_f = \text{unit sphere of } H_f.$$

We also denote H_f by $l_2(f)$ when $H = l_2$.

Theorem 3. *Every fixed point free involution on* S_f *is conjugate to the antipodal map.*

Corollary 1. *Let* X *be a space homeomorphic to* S *or* S_f*: Then any two fixed point free involutions on* X *are conjugate.*

In view of known results, X includes all Hilbert spaces H, all H_f and all separable infinite dimensional Fréchet spaces, etc.

The following are some of the main corollaries which give interesting symmetric versions of the results of Anderson [1], Wong [21] and Bessaga [3] in that order. First we let $s = \mathbb{R}^{\omega}$, the countable infinite product of reals $\mathbb{R}$. It is clear that s is a Fréchet space in its natural linear structure. Let $s_f = \{x(\cdot) \in s \colon x(i) = 0$ for almost all $i\}$.

Corollary 2. $l_2\backslash\{0\} \cong s\backslash\{0\}$, $l_2(f)\backslash\{0\} \cong s_f\backslash\{0\}$ and there is a homeomorphism h (with respect to each case) satisfying $h(x)+h(-x) = 0$ for all x.

Corollary 3. The separable Hilbert sphere is C^{∞} diffeomorphic to $l_2\backslash\{0\}$ by means of a diffeomorphism h such that $h(x)+h(-x) = 0$ for all x.

There is associated with an involution α on a space X the orbit space X/α. Denote by $P_{\alpha} : X \to X/\alpha$ the projection. If α is fixed point free, then P_{α} is a local homeomorphism. Suppose α_1, α_2 are fixed point free involutions on some Hilbert sphere S, then $P_{\alpha_1} = S/\alpha_1$ and $P_{\alpha_2} = S/\alpha_2$ are connected paracompact H-manifolds whose homotopy groups are identical on all dimensions and by elementary covering theory (see for example, Spanier [17-7.2.11]),

$$\pi_n(P_{\alpha_1}) = \pi_n(P_{\alpha_2}) = \begin{cases} \mathbb{Z}_2 \text{ (integers modulo 2)} & \text{for } n = 1 \\ 0 & \text{for } n \geq 2. \end{cases}$$

Theorem 4. Let α_1, α_2 be fixed point free involutions on S_0, where $S_0 = S$ or S_f. Then $S_0/\alpha_1 \cong S/\alpha_2$.

Moreover, if S is separable and α_1, α_2 are C^{∞}-smooth involutions on S, then (with respect to the natural C^{∞}-smooth structure on S/α_1 for which the projection $P_{\alpha_i} : S \to S/\alpha_i$ is smooth) S/α_1 is C^{∞}-diffeomorphic with S/α_2.

Non-trivial examples of involutions on s, H or Hilbert cube

$Q = [-1,1]^{\omega}$ are provided by the well known facts $s \cong s\times Q$, $H \cong H\times Q$ and the following theorem of West [19 - Corollary 5.1].

The product of a countable infinite collection of (non-degenerate) compact, contractible polyhedra is homeomorphic with the Hilbert cube, Q.

Let H be a Hilbert space and S the Hilbert sphere of H. Let $m_1 : H\setminus\{0\} \to S$ be any homeomorphism such that $m_1(S)$ = equator (S). Let $\gamma_* = m_1\gamma m_1^{-1}$, where $\gamma(x) = -x$ is the reflection on H. By theorem 1 there is a homeomorphism $m: (S,\gamma_*) \to (S,\gamma)$. Then let $mm_1(S) = S_0$, $A = mm_1(A_0)$ and $B = mm_1(B_0)$, where A_0, B_0 are respectively the bounded and unbounded components of $H\setminus S$. We then obtain the following non-trivial example in S. (Note that such an example trivially exists in H).

Example 1. *There is a bi-collared sub Hilbert sphere* $S_0 \subset S$ *such that* $S\setminus S_0 = A\cup B$, $A \cong H \cong B$ *and* S_0, A, B *are all symetric subsets of* S.

3. Preliminaries

We summarize some of the well known results which will be needed for proving the stated theorems in section 2.

Covering spaces. Two maps $f, g : X \to Y$ are homotopic relative $A \subset X$ provided there is a homotopy $\{\lambda_t\}$ joining f and g such that for each $a \in A$, $\lambda_t(a) = \lambda_0(a)$ for all t. In this case we write $f \sim g$ rel(A).

Proposition 1. *Let* $P: (E,e_0) \to (B,b_0)$ *be a covering space with base points. Then*

(1.1) *each path* $\sigma: [0,1] \to (B,b_0)$ *lifts to a unique path* $\sigma': [0,1] \to (E,e_0)$;

(1.2) *Suppose* $\sigma, \tau: [0,1] \to (B,b_0)$ *are loops in* (B,b_0) *such that* $\sigma \sim \tau$ *rel*(0,1), *then* σ, τ *lift to* σ', τ' *in* (E,e_0) *such that* $\sigma' \sim \tau'$ *rel*(0,1). *In particular, we have*

(1.3) *Let* c, c' *denote respectively the constant maps* $[0,1] \to (B,b_0)$ *and* $[0,1] \to (E,e_0)$. *Then* $\sigma \sim c$ *rel*$(0,1) \iff \sigma' \sim c'$ *rel*(0,1).

Infinite dimensional topology. In the following let M and N be connected paracompact manifolds modeled on a metric locally convex TVS F such that $F \cong F^\omega$ or F^ω_f, where F^ω is the countable infinite product of F and F^ω_f is the subspace in F^ω consisting of all points which have at most finitely many non-zero coördinates. Proofs of (2.1), (2.2) can be found in [10] and [12]. Proofs of (2.3) are contained in [13] and [4]. For a discussion and proof of (2.4), see for example, Eells and Elsworthy [8].

Proposition 2.

(2.1) (*open imbedding*) *There is an open imbedding* $h: M \to F$.

(2.2) (*classification*) *If* F *is a normed* TVS, *then each homotopy equivalence between* M *and* N *is homotopic to a homeomorphism*.

(2.2) $S \cong H\backslash\{0\} \cong H \cong H^\omega$.

Smooth classification

(2.4) *Every homotopy equivalence between separable C^∞-Hilbert manifolds is homotopic to a C^∞ diffeomorphism.*

4. Other results on infinite dimensional (I-D) spaces

For any map $h: X \to X$, let $fp(h) = \{x: h(x) = x\}$. The reflection map $x \to -x$ of any topological vector space (TVS) will be denoted by γ. If spaces X, Y carry involutions α, β respectively, then a map $m: (X,\alpha) \to (Y,\beta)$ induces a map $m': X/\alpha \to Y/\beta$. Obviously α is conjugate to β if and only if there is a homeomorphism $m: (X,\alpha) \to (Y,\beta)$. Let Q denote the Hilbert cube $[-1,1]^{\omega}$. A subset K of a TVS F is called symmetric if $-x \in K$ whenever $x \in K$.

If X is a space carrying an involution α and $Y \subset X$ is such that $\alpha(Y) \subset Y$, then we write (Y,α) for $(Y,\alpha|_Y)$.

If X is a space homeomorphic to $X \times F$, F a TVS, we define a subset K of X to be F-deficient provided there is a homeomorphism $h: X \to X{\times}F$ such that $h(K) \subset X{\times}\{0\}$. Originally V. Klee in [13] showed that any compact $K \subset l_2$ is l_2-deficient. It is known [5 - theorem 1] that for any manifold M modeled on a Fréchet space $F \cong F^{\omega}$, a closed $K \subset M$ is F-deficient if and only if K has property Z; that is, interior $(K) = \emptyset$ and for any homotopically trivial open $U \subset M$, $U \backslash K$ remains homotopically trivial. For instance, property Z is present if K is closed and locally compact.

Homeomorphism extension

Theorem 5. *Let A, B be a closed H-deficient subset of a Hilbert space H carrying involutions α, β respectively. Suppose $m: (A,\alpha) \to (B,\beta)$ is a homeomorphism, then there are involutions $\tilde{\alpha}$, $\tilde{\beta}$ on H extending α, β respectively and a homeomorphism $\tilde{m}: (H,\tilde{\alpha}) \to (H,\tilde{\beta})$ extending m such that $fp(\tilde{\alpha}) = fp(\alpha)$ and $fp(\tilde{\beta}) = fp(\beta)$.*

Corollary 4. *Let A be a closed H-deficient subset of a Hilbert space H. Then each involution α on A extends to an involution $\tilde{\alpha}$ on H such that $fp(\tilde{\alpha}) = fp(\alpha)$.*

First we need the following lemma.

Lemma 1. *Let Δ denote the diagonal of the product $H \times H$; that is $\Delta = \{(x,x) : x \in H\}$. Then there is an $h \in G(H{\times}H)$ such that $h(\Delta) = \{0\} \times H$.*

Proof. Let θ denote the countable clockwise $\pi/4$ rotation of the plane $\mathbb{R}^2$. Define $h: H\times H \to H\times H$ as follows. For any $(x(\cdot),y(\cdot)) \in H\times H$, $h(x(\cdot),y(\cdot)) = (\tilde{x}(\cdot),\tilde{y}(\cdot))$ where $(\tilde{x}(i),\tilde{y}(i)) = \theta(x(i),y(i))$ for any $i \in I(N)$.

Proof of theorem 5. First we remark that for any metric locally convex TVS $F \cong F\times F$, by a technique of V. Klee [13] any homeomorphism between two closed F-deficient subsets of F extends to one on F. So let $\tilde{m}$ be any homeomorphism of H extending m.

Let $\phi: H \to H\times H$ be a homeomorphism such that $\phi(A) \subset H\times\{0\}$. For any $a \in A$, let $\phi(a) = (a_1,0)$ and $\phi(\alpha(a)) = (a_2,0)$ and consider the following commuting diagram.

$$\begin{array}{ccc} A & \xrightarrow{\ m_1\ } & ((H\times H)\backslash K) \times (H\times H) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma_\Delta \times \gamma} \\ A & \xrightarrow{\ m_1\ } & ((H\times H)\backslash K) \times (H\times H) \end{array}$$

where $m_1(a) = (a_1,a_2) \times (0,0)$,

$K = \Delta\backslash K_1$, where K_1 is the projection of $m_1(fp(\alpha))$ onto the first two factors,

$\gamma_\Delta: H\times H \to H\times H$ such that $\gamma_\Delta(x,y) = (y,x)$ and

γ is the reflection on $H \times H$.

By lemma 1, K is a locally closed (that is, a difference of two closed sets), H-deficient subset of $H \times H$. Hence by Cutler [7 - theorem 1], $(H\times H)\backslash K \cong H$. Let $m_2: (H\times H)\backslash K \times (H\times H) \to H$ be a homeomorphism. Using the remark above we let λ denote a homeomorphism of H extending $m_2 m_1$. Denote the induced involution $m_2 \circ (\gamma_\Delta\times\gamma) \circ m_2^{-1}$ on H by μ. It is clear that $fp(\mu) = m_2(K_1)$. Let $\tilde{\alpha} = \lambda^{-1}\mu\lambda$. Then α is an involution on H extending α such that $fp(\tilde{\alpha}) = fp(\alpha)$. Let $\tilde{\beta} = \tilde{m}\tilde{\alpha}\tilde{m}^{-1}$. $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{m}$ fulfil the conditions of the theorem.

Remark. It is possible that theorem 1 of Cutler [7] also works for H_f. If such is the case we may replace H by H_f in theorem 5 and theorem 8 below.

Homeomorphism spaces are contractible

Let $G_0(X)$ denote the subspace of $G(X)$ consisting of all involutions on X. In the following let F be any TVS.

Theorem 6. *Let X be space which is homeomorphic to F^ω or F^ω_f. Then $G_0(X)$ is contractible.*

Proof. Say a space X has the reflexive isotopy property (RIP) ([18]) if the map $\phi: X^\omega \to X^\omega$ which takes each $(x_1,x_2,\ldots)$ to $(x_2,x_1,x_3,x_4,\ldots)$ is invertibly isotopic to the identity of X^ω. West [18 - corollary 2.3]shows that F has the (RIP). Renz in [16] shows that the space $G(X^\omega)$ is contractible provided X has the (RIP). Hence $G(F^\omega)$ is contractable. If, the construction of the contraction $\Phi(h,t)$ in [16 - 184], is replaced by

$$\Phi(h,t) = \phi^n(\cdot,t)^{-1} \circ h^{(n)} \circ \phi^{(n)}(\cdot,t) \;,$$

we still get a contraction of $G(F^\omega)$. In this case if $h \in G_0(F^\omega)$, then $h^{(n)}$ is induced by h and clearly a member of $G_0(F^\omega)$. Since $G_0(F^\omega)$ is invariant under conjugation, we have $\Phi(h,t) \in G_0(F^\omega)$ for each $(h,t) \in G_0(F^\omega) \times I$. Hence $G_0(F^\omega)$ is contractible. An inspection of Renz's proof shows that it works with F^ω replaced by F^ω_f.

Closed Imbeddings

Theorem 7. *Let X be a complete metric space carrying a fixed point free involution* α. *Then for some Hilbert space H and any fixed point free involution β on H, there is a closed imbedding* $m: (X,\alpha) \to (H,\beta)$.

In particular, there is a closed imbedding $m: X \to S$ *such that* $m(x) + m(\alpha(x)) = 0$ *for all* x.

Proof. It is known that for some Hilbert space H, there is a closed imbedding $m_1: X \to H\times H$ such that $m(X) \subset \{0\}\times H$. Let $\alpha_1 = m_1\alpha m_1^{-1}$ denote the induced involution on $m_1(X)$. By corollary 4 we can extend α_1 to a fixed point free involution β_1 on $H \times H$. By corollary 1 there is a homeomorphism m_2: $(H,\beta_1) \rightarrow H,\beta)$. Now let $m = m_2 m_1$. The second half of theorem 7 follows from the same consideration using $S \cong H$.

Theorem 8. *Suppose* M *is a connected* H-*manifold and* M, H *carry fixed point free involutions* α, β *respectively, then there is a closed imbedding* m: $(M,\alpha) \to (H,\beta)$ *such that* $m(M)$ *is a submanifold of* H.

(See remark following theorem 5.)

Proof. By Henderson-West [11 - theorem 6], M can be imbedded as a closed submanifold of $H \times H$ which is contained in $\{0\} \times H$. The rest of the proof is the same as theorem 7.

We may regard $Q(= [-1,1]^{\omega})$ as a subset of $s = \mathbb{R}^{\infty}$. In particular, the reflection $\gamma(x) = -x$ on s induces a reflection on Q also denoted by γ.

Theorem 9. *Let* α *be an involution on* X, $X = Q$ *or* s. *Then there is a map* m: $(X,\alpha) \to (X,\gamma)$ *such that* $m|_{X \backslash fp(\alpha)}$ *is an imbedding. Moreover, if* $X = Q$ *and* α: $Q \to Q$ *has exactly one fixed point, then* m *is an imbedding.*

Remark. I have known of a (rather) geometrical argument for this theorem. However, the following simple but elegant proof is provided by W. Barit.

Proof. Define a function m: $X \to X$ in the following manner. For any $x = (x_1,x_2,\ldots) \in X$, denote $\alpha(x) = (y_1,y_2,\ldots) \in X$. We can write $x = X_1 \times X_2 \times \ldots$, where each $X_n = X$. Define $m(x) = (z_1,z_2,\ldots)$ by

$$z_1 = \tfrac{1}{2}(x_1-y_2),\ \tfrac{1}{2}(x_2-y_2),\ \ldots) \qquad \text{and for } n > 1,$$

$$z_n = (\tfrac{1}{4}(x_n-y_n)\ (x_1+y_1),\ \tfrac{1}{4}(x_n-y_n)\ (x_2+y_2),\ \ldots)$$

where $z_n \in X_n$.

Since X is convex and symmetric, each z_n is indeed a member of X. It is clear that m is continuous and is a map of (X,α) into (X,γ). Let us now show that m is one-to-one off $fp(\alpha)$. First of all we easily observe that $m(x) = 0$ if and only if $x \in fp(\alpha)$. So suppose $m(x) = m(x')$ for some $x, x' \notin fp(\alpha)$. Write $x' = (x_1', x_2', \ldots)$, $\alpha(x') = (y_1', y_2', \ldots)$ and $m(x') = (z_1', z_2', \ldots)$. Let $n \geq 1$ be fixed. Since $x \notin fp(\alpha)$, $x \neq \alpha(x)$. Hence $x_i - y_i \neq 0$ for some i. Since $z_1 = z_1'$ and $z_n = z_n'$, we have $x_n - y_n = x_n' - y_n'$ and $(x_i - y_i)(x_n + y_n) = (x_i' - y_i')(x_n' + y_n')$. But $x_i - y_i = x_i' - y_i'$, hence $x_n + y_n = x_n' + y_n'$. This implies $x_n = x_n'$. So m is one-to-one off $fp(\alpha)$. An inspection shows that local inverse exists for any $m(x) \neq 0$. So $m|_{X \setminus fp(\alpha)}$ is an imbedding. Finally if $X = Q$ and α has exactly one fixed point, then m is one-to-one. Since Q is compact, m is an imbedding.

Negligible Subsets

Theorem 10. Let $K_1, K_2, \ldots$ be closed H-deficient subsets of a Hilbert space H. Suppose H carries fixed point free involutions α and β such that $\alpha(\bigcup_{i>1} K_i) = \bigcup_{i>1} K_i$, then there is a homeomorphism m: $(H \setminus \bigcup_{i \geq 1} K_i, \alpha) \to (H, \beta)$.

Theorem 11. Let X be a space homeomorphic to s, l_2 or the unit sphere S of l_2. Let $K_1, K_2, \ldots$ be closed symmetric Z-sets in X such that $0 \notin K_i$ for all i. Let γ denote the reflection on X ($\gamma|_S$ = antipodal map). Then there is a homeomorphism m: $(X \setminus \bigcup_{i \geq 1} K_i, \gamma) \to (X, \gamma)$.

Proof of theorem 10. By Cutler ([7-Theorem 1]) $H \setminus \bigcup_{i \geq 1} K_i \overset{m_1}{\cong} H$. $\alpha_1 = m_1 \alpha m_1^{-1}$. By corollary 1 $(H, \alpha_1) \overset{m_2}{\cong} (H, \beta)$. Let $m = m_2 m_1$.

Proof of theorem 11. We will indicate a proof for X = s. A proof for $X = l_2$ or S can be carried out in exactly the same manner. Write $s = \mathbb{R}^\omega$ as $\mathbb{R}_1 \times \mathbb{R}_2 \times \ldots$ and denote by s_n the subset $\{(0, \ldots, 0)\} \times \mathbb{R}_{n+1} \times \mathbb{R}_{n+2} \times \ldots$ of $s (s_0 = s)$. Let $K = \bigcup_{i \geq 1} K_i$. Clearly $K = \bigcup_{n \geq 0} ((s_n \setminus s_{n+1}) \cap K)$. Let $L_n = (s_n \setminus s_{n+1}) \cap K$ and let

$$s_n^+ = \{(x_i) \in s_n : x_{n+1} \in (0, \infty)\},$$

$$s_n^- = \{(x_i) \in s_n : x_{n+1} \in (-\infty, 0)\}.$$

By known results $K \cap s_0^+$ can be strongly extracted from s_0^+. Hence there is a homeomorphism $f_1: s\backslash(K\cap s_0^+) \to s$ such that $f_1|_{s\backslash s_0^+}$ is the identity. Define $m_1: (s\backslash L_1,\gamma) \to (s,\gamma)$ by $m_1(x) = f_1(x)$ if $x \in s\backslash(s_0^+ \cup K)$ and $m_1(x) = -f_1(-x)$ otherwise. (Note that the condition that L_1 be symmetric is neccessary here.) By similar considerations there is a homeomorphism (for $n\geq 1$)
$m_n: (s\backslash L_n,\gamma) \to (s,\gamma)$ such that m_n leaves the first $(n-1)$ coördinates of each point unchanged and m_n may be limited by any open cover of s. By the convergence procedure of Anderson-Bing [2] we can require that $m_n \ldots m_2 m_1$ converges to a homeomorphism $m: (s\backslash K,\gamma) \to (s,\gamma)$.

Using this result and the homeomorphism $S \to s$ defined in Anderson-Bing [2 - lemma 3.2], we can give a proof of the following theorem independent of results in section 2.

Corollary 5. Let S be the unit sphere of l_2. Then there is a homeomorphism $m: S \to s\backslash\{0\}$ such that $m(x)+m(-x) = 0$ for all x.

Proof. By theorem 11 there are homeomorphisms
$m_1: (S,\gamma) \to (S\backslash \bigcup_{i\geq 1} (E^i \cap S),\gamma)$ and $m_2: (s\backslash(\bigcup_{i\geq 1} (C_i\backslash\{0\})),\gamma) \to (s,\gamma)$, where E^i is the i-dimensional Euclidean subspace in l_2 and
$C_i = \{(X_n) \in s: \sum x_n^2 \leq i$ and for each $n > 0$, $|x_n| \leq 1 - 1/i\}$.
(Clearly each C_i is a compact symmetric subset of s.) An inspection of the homeomorphism $h: S\backslash \bigcup_{i\geq 1} (E^i \cap S) \to s\backslash \bigcup_{i\geq 1} C_i$ defined in lemma 3.2 of [2] shows that h is in fact a homeomorphism of $(S\backslash \bigcup_{i\geq 1} E^i,\gamma) \to (s\backslash \bigcup_{i\geq 1} C_i,\gamma)$.

Then $m = m_2 h m_1$ is as required.

5. Lemmas

For any Hilbert sphere S of $l_2(N)$, we can speak of the North-South poles and equator of S in the following sense. Give the index set $I(N)$ any well-ordering and denote its first member by a_0. Then denote $0_2 = x(\cdot) \in S$ such that $x(a_0) = 1$, $0_1 = x(\cdot) \in S$ such that $x(a_0) = -1$ and the equator $S_0 = \{x(\cdot) \in S: x(a_0) = 0\}$. We assume each Hilbert sphere is equipped with a fixed $\{0_1, 0_2, S_0\}$.

Lemma 2. *Let $A: S \to S$ be the antipodal map on a Hilbert sphere S and an arbitrary fixed point free involution on S. Suppose S/A is homeomorphic with S/α by an homeomorphism $m_0: S/A \to S/\alpha$, then m_0 induces a homeomorphism $m_0: S \to S$ so that the following diagram commutes.*

$$\begin{array}{ccc} S & \xrightarrow{\;m\;} & S \\ \downarrow p & & \downarrow p_\alpha \\ S/A & \xrightarrow{\;m_0\;} & S/\alpha \end{array}$$

where p, p_α are projection maps.

Proof. The mappings p and p_α are universal coverings. Two universal coverings of homeomorphic bases are necessarily isomorphic.

If in lemma 2 we let α be a C^∞-smooth involution on S, the orbit space S/α has a naturally induced C^∞-smooth structure making S/α a smooth manifold such that the projection $p_\alpha: S \to S/\alpha$ is C^∞-smooth. We have the following lemma.

Lemma 3. *Let S be a Hilbert sphere and let α be a C^∞-smooth involution on S. Suppose S/A is C^∞- diffeomorphic with S/α with a diffeomorphism $m_0: S/A \to S/\alpha$, then m_0 induces an C^∞-diffeomorphism $m: S \to S$ such that $p_\alpha m = m_0 p$.*

Given an (infinite dimensional) Hilbert sphere S we may assume S contains the separable Hilbert sphere S_0 which is the unit sphere of l_2. We may suppose the North-South poles of S are the same as those for S_0 and are the points $0_2 = (1,0,0,\ldots)$ and $0_1 = (-1,0,0,\ldots)$. By the n-sphere in S we mean the standard n-sphere, S^n, in l_2. Let A: S → S denote the antipodal map and denote $A|_{S^n}$ by A_n. Let S^∞ denote the inductive limit $\varinjlim S^n$; that is $S^\infty = \bigcup_{n=1}^{\infty} S^n$ and is equipped with the coherent (weak) topology (see Palais [15 - page 4]). Similarly let P^∞ denote the inductive limit $\varinjlim P^n$, where $P^n = S^n/A_n$. It follows from application of the covering maps $S^n \to p^n$, that the homotopy groups of P^∞ are $\{Z_2, 0, 0, \ldots\}$ (see Spanier [16 - 7.2.11]).

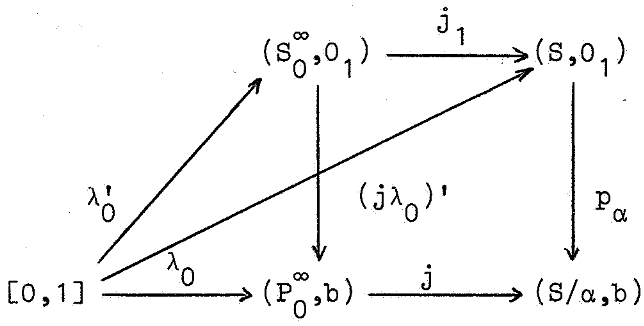
Now let α be any fixed point free involution on S such that $\alpha(0_1) = 0_2$. Suppose there is a map h: $S^\infty \to S$ such that $h(0_1) = 0_1$, $h(0_2) = 0_2$, $\alpha h(x) = h(-x)$ for all $x \in S^\infty$ and $h|_{S^n}$ is an imbedding for each n, then we can give the set $h(S^\infty) = \bigcup_{n=1}^{\infty} h(S^n)$ the coherent topology and denote this space by S_0^∞. Then $h: S^\infty \to S_0^\infty$ is a homeomorphism which induces a homeomorphism $h_0: P^\infty \to P_0^\infty$ where $P_0^\infty = \varinjlim h(S^n)/\alpha_n$, $\alpha_n = \alpha|_{h(S^n)}$. Let p_α: S → S/α denote the projection. Note the induced projection $S_0^\infty \to P_0^\infty$ is continuous and will be denoted also by p_α. P_0^∞ is naturally included in S/α and we assert

<u>Lemma</u> 4. <u>The inclusion</u> j: $P_0^\infty \to S/\alpha$ <u>is a homotopy equivalence</u>.

<u>Proof</u>. Since P_0^∞ is a CW-complex and S/α is dominated by CW-complex [15 - theorem 14], by a well known theorem of Whitehead [20 - theorem 1], it suffices to show j is a weak homotopy equivalence. Since the homotopy groups of P_0^∞ and S/α are identically equal $\{Z_2, 0, 0, \ldots\}$, we need only to verify that $j_*: \pi_1(P_0^\infty) \to \pi_1(S/\alpha)$ is an isomorphism. Let $b = p_\alpha(0_1)$ be the base point for P_0^∞ and S/α and let λ_0: [0,1] → (S/α,b) rel(0,1) be a map belonging to the non-trival member in $\pi_1(S/\alpha)$. Denote the lifting of λ_0 into S by λ_0': [0,1] → (S,0_1). Since S has a linear structure, namely S ≅ H, by proposition 1 the end point $\lambda_0'(1) = 0_2$. Now let g: [0,1] → S^∞ be any map such that $g(0) = 0_1$, $g(1) = 0_2$ and let $\lambda_1' = hg$. Then $\lambda_1' \sim \lambda_0'$ rel(0,1) by the linear structure on S. Let $\lambda_1 = p_\alpha \lambda_1'$. Hence $\lambda_1 \sim \lambda_0$ rel(0,1) and $j_*([\lambda_1]) = [\lambda_0]$. We have shown j_* is onto.

Next suppose λ_0: [0,1] → (P_0^∞,b) rel(0,1) is a map such that $j\lambda_0 \sim c$ rel(0,1) in S/α, where c: [0,1] → (S/α,b) is the constant map. Let λ_0' denote the lifting of λ_0 into S_0^∞, $(j\lambda_0)'$ the lifting of $j\lambda_0$ into S and j_1: $S_0^\infty \to S$ the inclusion (j_1 is continuous). Then $j_1\lambda_0'$

is another lifting of $j\lambda_0$ into S. By uniqueness (proposition (1.1)) we have $j_1\lambda_0' = (j\lambda_0)'$ and the following diagram commutes



Since $j\lambda_0 \sim c$, $(j\lambda_0)'(0) = 0_1 = (j\lambda_0)'(1)$. Thus $\lambda_0'(0) = 0_1 = \lambda_0'(1)$. But $S_0^\infty \cong S^\infty$ and $S^\infty \setminus \{\text{point}\}$ can be given a linear structure. Namely, $S^\infty \setminus \{\text{point}\} \simeq E^\infty$, where $E^\infty = \varinjlim E^n$, E^n the n-dimensional Euclidean space in l_2. Hence $\lambda_0' \sim c_1$ rel(0,1), where c_1: $[0,1] \to (S_0^\infty, 0_1)$ is the constant map. So $[\lambda_0]$ is the zero member in $\pi_1(P_0^\infty)$ and j_* is one-to-one. The proof of lemma 4 is now complete.

<u>Lemma</u> 5. <u>Let</u> f: K → M <u>be a piecewise linear map of a finite simplicial complex</u> K <u>into</u> M, <u>an open subset of some infinite dimensional Hilbert space</u> H. <u>Then for any subcomplex</u> L ⊂ K, <u>there is a piecewise linear map</u> g: K → M <u>such that</u> (1) $g|_L = f$, (2) $g(K\setminus L) \cap f(L) = \emptyset$, (3) g <u>is one-to-one on</u> $K\setminus L$ <u>and</u> (4) $g \sim f$ rel(L).

<u>Proof</u>. This is a routine application of the linear structure together with the infinite dimensionality of M.

6. The Key Lemma

Lemma 6. Let α be a fixed point free involution on a Hilbert sphere S such that $\alpha(0_1) = 0_2$ and let $S^\infty = \varinjlim S^n$ be defined as in section 5. Then there is a map $h: S^\infty \to S$ such that $h(0_1) = 0_1$, $h(0_2) = 0_2$, $\alpha h(x) = h(-x)$ for all $x \in S^\infty$ and $h|_{S^n}$ is an imbedding for all $n \geq 0$.

Proof. Since S/α is an H-manifold, we may assume by proposition (2.1) that S/α has been imbedded as an open subset of H. We shall construct h by induction on the spheres S^n. So suppose that for some $n \geq 0$ there is a map $h_n: S^n \to S$ satisfying all the properties of the lemma and such that the map $p_\alpha \cdot h_n: S^n \to S/\alpha$ is piecewise linear. h_0 exists trivially. We want to construct an $h_{n+1}: S^{n+1} \to S$ which extends h_n. First of all let $f: S_+^{n+1} \to S$ be any map which extends h_n, where S_+^{n+1} is the closed upper half sphere of S^{n+1}. Then $p_\alpha \cdot f: S_+^{n+1} \to S/\alpha$ extends $p_\alpha \cdot h_n$. We may approximate $p_\alpha \cdot h_n$ by a piecewise linear one, and by the statements of Lemma 5 there is a piecewise linear map $g: S_+^{n+1} \to S/\alpha$ such that (1) $g|_{S^n} = p_\alpha \cdot h_n$, (2) $g(S_+^{n+1} \backslash S^n) \cap p_\alpha \cdot h_n(S^n) = \emptyset$, (3) g is one-to-one on $S_+^{n+1} \backslash S^n$ and (4) $g \sim p_\alpha \cdot f \operatorname{rel}(S^n)$.

Let $b = p(0_1)$ denote the base point for S/α. For each $x \in S_+^{n+1} \backslash \{0_1, 0_2\}$ let $S(x)$ denote the unit sphere of the two dimensional subspace generated by x, 0_1 and 0_2 and let $C(x)$ denote the arc in $S(x) \cap S^{n+1}$ joining 0_1 and x. The map $g|_{C(x)}: C(x) \to (S/\alpha, b)$ lifts to a map $g_1: C(x) \to (S, 0_1)$ such $g_1(0_1) = 0_1$. Let x' denote the end point $g_1(x)$ of the arc $g_1(C(x))$. Define $h_{n+1}: S^{n+1} \to S$ by

$$h_{n+1}(x) = \begin{cases} 0_1 & \text{if } x = 0_1, \\ 0_2 & \text{if } x = 0_2, \\ x' & \text{if } x \in S_+^{n+1} \backslash \{0_1, 0_2\} \text{ and} \\ \alpha(-x') & \text{otherwise.} \end{cases}$$

We assert that h_{n+1} satisfies all the properties of the lemma.

First of all if $x \in S^n \backslash \{0_1, 0_2\}$, then since by (1) $g|_{S^n} = p_\alpha \cdot h_n$, the map $h_n|_{C(x)}$ is another lifting of the map $p_\alpha \cdot h_n|_{C(x)} = g|_{C(x)}$. Hence $h_n|_{C(x)} = g_1$ and it follows that $h_{n+1}|_{S^n} = h_n$. Next we assert that $h_{n+1}|_{S_+^{n+1}}$ is continuous. For $x \in S^{n+1}\backslash\{0_1,0_2\}$, let $\tilde{C}(x)$ denote the half circle in $S(x) \cap S^{n+1}$ containing x. We claim that the map $g|_{\tilde{C}(x)}$ must lift to a map $\tilde{g}$: $\tilde{C}(x) \to S$ such that $\tilde{g}(0_1) = 0_1$, $\tilde{g}(0_2) = 0_2$. If not, then $\tilde{g}_1(0_1) = 0_1 = \tilde{g}_1(0_2)$ and it implies $g|_{\tilde{C}(x)} \sim c \ \mathrm{rel}(0_1,0_2)$, where c: $\tilde{C}(x) \to (S/\alpha,b)$ is the constant map. Recall that we have started with a map f: $S_+^{n+1} \to S$ which extends h_n. Hence $f(0_1) = 0_1$ and $f(0_2) = 0_2$ and $p_\alpha \cdot f|_{\tilde{C}(x)}$ is not homotopic to c $\mathrm{rel}(0_1,0_2)$. By property (4) of g we arrive at a contradiction.

Following from the above discussion and the uniqueness of the lifting g_1, we have a well-defined continuous function h_{n+1}: $S_+^{n+1} \to S$. By the very definition of h_{n+1}, h_{n+1}: $S^{n+1} \to S$ is indeed an extension of h_n: $S^n \to S$ and satisfies $\alpha h_{n+1}(x) = h_{n+1}(-x)$ for all $x \in S^{n+1}$.

Finally it follows from properties (2) and (3) of g we have $h_{n+1}(S_+^{n+1}) \cap \alpha h_{n+1}(S_+^{n+1}) = h_{n+1}(S^n)$. Thus h_{n+1} is an embedding of S^{n+1} into S. The map h: $S^\infty \to S$ such that $h|_{S^n} = h_n$ fulfills all the requirements of the lemma.

7. Proofs of theorems 1 - 4

Proof of theorem 1. Let A: $S \to S$ denote the antipodal map and α any fixed point free involution on S. Without loss of generality we may assume $\alpha(0_1) = 0_2$. Let P^∞ denote the inductive limit $\varinjlim P^n$ as defined in section 5. By the results of lemma 6 and the discussion in section 5, we have a corresponding inductive limit for α, namely $P_0^\infty = \varinjlim h(S^n)/\alpha^n$, where h: $S^\infty \to S$ is given by lemma 6. By lemma 4 the inclusions $P^\infty \to S/A$ and $P_0^\infty \to S/\alpha$ are homotopy equivalences. The homeomorphism h also induces a homeomorphism $P^\infty \to P_0^\infty$. Hence S/A and S/α have the same homotopy type and by proposition (2.2), $S/A \cong S/\alpha$. The statetement of lemma 2 shows that there is a homeomorphism m: $(S,A) \to (S,\alpha)$.

Proof of theorem 2. The same argument as above shows that S/A and S/α have the same type. By proposition (2.4) they are C^∞ diffeomorphic. Now apply lemma 3.

Proof of theorem 3. The proof is exactly the same as the one for theorem 1. Note that in this case we also have a similar classification theorem for H_f-manifolds, namely, use corollary 1 and theorem 11 of [11].

Proof of corollary 1. Clear.

Proof of corollary 2. Let m: $l_2\backslash\{0\} \to s\backslash\{0\}$ be any homeomorphism. Let $\gamma_* = m^{-1}\gamma m$ denote the induced fixed point free involution on $l_2\backslash\{0\}$ by the reflection $\gamma(x) = -x$ on $s\backslash\{0\}$. Let m_1: $(l_2\backslash\{0\},\gamma) \to (l_2\backslash\{0\},\gamma_*)$ be any homeomorphism given by corollary 1. Then $h = mm_1$ is as required.

Proof of corollary 3. The same as above, using theorem 2 and the following result of Bessaga ([3]). S is C^∞ diffeomorphic with $l_2\backslash\{0\}$.

Proof of theorem 4. Same as those indicated in the proofs of theorem 1 - 3.

8. Open questions

For any Hilbert space H, there is a natural involution on H, namely the reflection $\gamma(x) = -x$. Such an involution has exactly $\{0\}$ as fixed point and the orbit space H/γ is locally H except at the image of the origin. Let $p: H \to H/\gamma$ denote the projection.

Question 1. *Suppose α is an involution on H with exactly $\{0\}$ as fixed point, is α conjugate to γ?*

In particular we can ask

Question 2. *Let α be as above and $0'$ the image of 0 under the projection $H \to H/\alpha$. Does there exist a homeomorphism h of H/α onto H/γ such that $h(0') = p(0)$?*

It seems likely that an affirmative answer for question 2 will settle question 1 affirmatively.

Question 3. *Let γ denote the reflection on the Hilbert cube Q and α any involution on Q with exactly $\{0\}$ as fixed point. Is α conjugate to γ ?*

Question 4. *Is l_2 homeomorphic to s by an h such that $h(x) + h(-x) = 0$ for all x ?*

Example 2. The function m defined in the proof of theorem 6 for $X = s$ and for $\alpha: s \to s$ having exactly one fixed point is one-to-one and continuous but not an imbedding. The following is a counter example.

Denote the space $(-1,1)^\omega$ (subset of $Q = [-1,1]^\omega$) by s_0 ($\cong s$). Let $X = s_0 \cup \{b\}$, where $b \in Q\backslash s_0$, and let $a \in s_0$. It is known that $X \cong s_0$ and there is an involution α on X such that $fp(\alpha) = \{a,b\}$ (see for example West [18]). Then $\alpha|_{s_0}$ is an involution on s_0 with exactly $\{\alpha\}$ as fixed point. The function m obviously takes a sequence that converges to $\{b\}$ (not in s_0) to a sequence that converges to 0. So $m^{-1}: m(s_0) \to s_0$ fails to be continuous. We ask

Question 5. Let α be an involution on s (or H) with exactly one fixed point. Is there an imbeddeing m: $(s,\alpha) \to (s,\gamma)$?

Question 6. Let M be a connected H-manifold. Suppose M, H carry fixed point free involutions α, β respectively. Is there an open imbedding m: $(M,\alpha) \to (H,\beta)$?

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